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2003 J. Phys. A: Math. Gen. 36 10615

(http://iopscience.iop.org/0305-4470/36/42/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 10615–10635 PII: S0305-4470(03)59490-2

Painlevé equations from Darboux chains: I. P_{III}–P_{V}

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Received 7 February 2003, in final form 14 July 2003
Published 7 October 2003
Online at stacks.iop.org/JPhysA/36/10615

Abstract

We show that the Painlevé equations P_{III}–P_{VI} can be derived in a unified way from a periodic sequence of Darboux transformations for a Schrödinger problem with quadratic eigenvalue dependence. The general problem naturally divides into three different branches, each described by an infinite chain of equations. The Painlevé equations are obtained by closing the chain periodically at the lowest nontrivial level(s). The chains provide 'symmetric forms' for the Painlevé equations, from which Hirota bilinear forms and Lax pairs are derived. In this paper (part I) we analyse in detail the cases P_{III}–P_{V}, while P_{VI} will be studied in part II.

PACS number: 02.30.Ik

1. Introduction

The aim of this paper is to provide a bridge between two extremely fruitful descriptions of the Painlevé equations in the context of integrable systems: the (Darboux) dressing chain approach of the Russian school [1–6] and the tau-function based approach of the Japanese school [7–16].

In the seminal paper [3] Adler proposed a complete description of the Painlevé equations $P_{\mathrm{II–VI}}$ by means of the so-called Darboux chains. These generalize the usual dressing chain [1] which, in its periodic case, was already known to provide a description of the P_{IV} and P_{V} equations [2]. Adler succeeded in describing the Schlesinger–Bäcklund transformations and notably the Weyl-group action of these transformations for the Painlevé equations, merely by starting from the Darboux transformations that underlie their associated (periodic) chain equations. However, its relationship with Okamoto's theory of the Painlevé equations (in which Weyl-groups play a central role) has still not been elaborated. In the Okamoto description, the Weyl-group action of the Bäcklund transformations for the Painlevé equations is closely related to the notion of a *tau-function* [7, 9–13] (see [17] for an overview), which in turn is connected to the existence of isomonodromy problems associated with the Painlevé equations [8]. In recent years this (affine) Weyl-group structure of certain Bäcklund transformations for the

Painlevé equations, together with the tau-function description of the Hamiltonians underlying these equations (see [18] for a survey of results), has given rise to a 'symmetry-based' approach to the Painlevé equations and related systems [14, 19]. In [14] Noumi and Yamada proposed a systematic description of a class of dynamical systems comprising the P_{IV} and P_{V} equations, which all possess Bäcklund transformations that make up an affine Weyl-group and which allow for tau-function descriptions of their Hamiltonian structures. The so-called 'symmetric forms' of these dynamical systems play a central role in this approach [16].

We will show, using the associated Hirota bilinear representations, that periodic closings of the standard dressing chain considered in [2, 3, 6]—and from which $P_{\mathrm{IV\text{–}V}}$ are obtained at periods 3 and 4 respectively—correspond exactly to the dynamical systems introduced in [14]. Our approach is, however, not restricted to the $P_{\mathrm{IV\text{–}V}}$ equations or similar cases. We shall also explicitly derive a symmetric form of the P_{III} equation and of a whole class of related equations (which first appeared in [4]), which we then bilinearize (i.e. express in terms of tau-functions). We believe that the bilinearization of these equations and of the P_{III} equation in particular is new.

A major part of this paper is devoted to a systematization of the results obtained by Adler [3] in order to derive all relevant Darboux (dressing) chains from a common starting point and to analyse their properties using one and the same technique. This is in contrast to what is presently available in the literature, where the nature of the Darboux chain (or the techniques used in its construction) is adjusted to each Painlevé equation specifically. This unified approach then not only pays off when bilinearizing the Darboux chains we obtain, but also and especially so when deriving Lax representations of those chains. Although a lot of attention has been devoted to the spectral properties of the linear equations whose Darboux transformations underlie the various (Darboux) dressing chains associated with the Painlevé equations [2, 6, 20], no Lax representations of these chains have been offered in the literature pertaining to Darboux chains, although the Noumi–Yamada symmetry approach does yield such representations. In the following we shall, systematically, derive Lax pairs for the various Darboux chains presented throughout this paper. In the $P_{\mathrm{IV\text{–}V}}$ case these Lax pairs will turn out to be identical to those obtained in [16, 21], but we believe that the result concerning P_{III} is new.

The structure of the paper is as follows: we start by defining the Darboux transformation for a Schrödinger operator with an energy-dependent potential and derive its associated chain equation. The analysis naturally divides into three branches covering P_{IV}–P_{V}, P_{III} and P_{VI}, respectively. In section 3 we study the P_{IV}–P_{V} branch in general and then in section 4 give the details for P_{IV} and in section 5 for P_{V}. Then we study the P_{III} branch in general and in section 7 give the specific details for P_{III} itself. As mentioned above, the results concerning the P_{VI} case will be discussed in part 2.

While completing this paper we came across [4] by Shabat, which is also concerned with systematizing the Darboux–Painlevé connection. It is therefore not surprising that our analysis runs largely parallel with that paper and that many formulae are identical or similar. However, as we wish to focus on the specific connections to the Painlevé equations, continuing with their bilinearization and Lax pairs, our presentation is both more specific and more detailed. We believe that in doing so the method and its internal machinery gain further clarity.

2. The Darboux transformation

Roughly speaking, a Darboux transformation is a transformation between two linear (ordinary or partial) differential equations, of the same overall appearance but for different values of their coefficients: the solutions of the first equation will be mapped to the solutions of the second,

and the changes in the coefficients of the equation are also provided by the transformation (see, e.g., [22] for a classic introduction in the context of integrable systems). We are interested in *chains* of such transformations applied to the Schrödinger equation.

2.1. The general form of the chain equations

Consider the spectral problem associated with the Schrödinger operator [4]

$$L_j(u,\lambda)\psi_j(\lambda,x)=0 \qquad \text{where} \qquad L_j(u,\lambda):=\partial_x^2+u_j(\lambda,x). \tag{1}$$

Here j indexes the eigenproblem, and λ are the eigenvalues (assumed non-degenerate). Note that we make no assumptions about boundedness or normalizability of the eigenfunctions. We shall refer to the functions $u_j(\lambda,x)$ as 'generalized potentials' as they include dependence on the eigenvalues λ. Next define the operator

$$G_j(\lambda,x):=A_j(\lambda,x)(\partial_x-F_j(\lambda,x)) \tag{2}$$

and then using this define new functions

$$\psi_{j+1}(\lambda,x):=G_j(\lambda,x)\psi_j(\lambda,x) \tag{3}$$

for each eigenfunction $\psi_j(\lambda,x)$ of the original problem at level j. The new functions ψ_{j+1} will then be eigenfunctions of a new operator L_{j+1} (with the *same, generic* eigenvalue λ), provided that L_{j+1} satisfies the operator identity

$$L_{j+1}G_j(\lambda,x)=\tilde{G}_j(\lambda,x)L_j(u,\lambda) \tag{4}$$

for some $\tilde{G}_j(\lambda,x):=A_j(\lambda,x)(\partial_x-\tilde{F}_j(\lambda,x))$. Assuming that L_{j+1} is a differential operator (i.e., polynomial in ∂_x) one finds that

$$\tilde{F}_j(\lambda,x)=F_j(\lambda,x)-2(\log A_j(\lambda,x))' \tag{5}$$

$$L_{j+1}=L_j(u_{j+1},\lambda) \tag{6}$$

$$u_{j+1}(\lambda,x)=u_j(\lambda,x)+[2F_j(\lambda,x)A_j(\lambda,x)-A_j(\lambda,x)']'/A_j(\lambda,x) \tag{7}$$

$$F_j(\lambda,x)'+F_j(\lambda,x)^2+u_j(\lambda,x)=\mu_j(\lambda)A_j(\lambda,x)^{-2} \tag{8}$$

where μ_j is an integration constant (the $'$-notation stands for $\frac{\mathrm{d}}{\mathrm{d}x}$).

> The transformation (3) from ψ_j to ψ_{j+1} will be called a *Darboux transformation* iff at each step j the operator G_j is such that *it annihilates some chosen eigenfunction* φ_j of (1) having eigenvalue ν_j.

This necessarily implies that $F_j(\nu_j,x)=(\log\varphi_j)_x$ and therefore these F_j also satisfy the equation

$$F_j(\nu_j,x)'+F_j(\nu_j,x)^2+u_j(\nu_j,x)=0. \tag{9}$$

This should be compatible with (8) and therefore $\mu_j(\nu_j)=0$.

Now subtracting (8) for j and for $j+1$ and using (7) to eliminate the potentials u we get a chain of equations

$$\begin{aligned}F_{j+1}(\lambda,x)'+(F_j(\lambda,x)-(\log A_j(\lambda,x))')'+F_{j+1}(\lambda,x)^2-(F_j(\lambda,x)-(\log A_j(\lambda,x))')^2\\ +\mu_j(\lambda)A_j(\lambda,x)^{-2}-\mu_{j+1}(\lambda)A_{j+1}(\lambda,x)^{-2}=0.\end{aligned} \tag{10}$$

In the following we shall think of this *chain equation* as a generator for λ-independent chain equations (the *dressing chains*). Thus the main equations underlying our analysis will be (7), which gives the change in u once F, A are given, and (10), the chain equation.

2.2. Linear problem for the chain equations

The commutation relation (4) which generates the chain equation (10) can also be rewritten as

$$L_{j+1}G_j - G_jL_j = 2A_j'L_j \qquad \text{or} \qquad A_j^2L_{j+1}G_j = G_jA_j^2L_j \tag{11}$$

which allows one to interpret the system (1), (3) as a kind of linear problem for the chain equation. Let us define

$$M_j := \partial_x - A_j^{-1}\mathcal{S} - F_j \qquad \text{where} \qquad \mathcal{S} : \psi_j(\lambda, x) \mapsto \psi_{j+1}(\lambda, x) \tag{12}$$

then since

$$M_j\psi_j(\lambda, x) = 0 \tag{13}$$

(i.e., equation (3)) we can eliminate ∂_x from the eigenvalue problem (1) and obtain the following second-order difference equation for the eigenfunctions $\psi_j(\lambda, x)$:

$$L_j^d\psi_j(\lambda, x) = 0 \tag{14}$$

where

$$L_j^d := A_j^{-1}\mathcal{S}A_j^{-1}\mathcal{S} + A_j^{-1}\mathcal{S}F_j + F_jA_j^{-1}\mathcal{S} - (\log A_j)'A_j^{-1}\mathcal{S} + \mu_jA_j^{-2} \tag{15}$$

(see also [4, 23, 24]). The compatibility condition of the system (13), (14), which is of course a natural consequence of the compatibility of equations (1) and (3), takes the form

$$A_j\left(M_jL_j^d - L_j^dM_j\right) + 2A_j'L_j^d = 0 \tag{16}$$

and is satisfied iff (10) holds. This linear problem will play an important role later on when we derive explicit forms for the Lax pairs for various reductions of the dressing chains.

2.3. The specialization that contains P_{III}–P_{VI}

The above is as far as we will go with the general setting. Now we specialize to the λ-dependence [3]

$$u_j(\lambda, x) = -\lambda^2 + \lambda v_j(x) + w_j(x) \tag{17}$$

for the potential, and

$$F_j(\lambda, x) = \lambda h_j(x) + f_j(x) \tag{18}$$

for the Darboux transformation. We will now show that this simple case is still rich enough to allow for three different infinite hierarchies of equations, containing Painlevé equations as their simplest nontrivial members.

With the above assumptions we find from (5) and (8) that both $(\log A_j(\lambda, x))'$ and $A_j(\lambda, x)^{-2}$ should be polynomials in λ. This is only possible if the λ-dependence in $A_j(\lambda, x)$ is multiplicative, in which case it can always be incorporated into the integration constants $\mu_j(\lambda)$ in (8). Furthermore, comparing (8) and (9) shows that $\mu_j(\lambda)$ should contain an overall factor of $\lambda - v_j$ and thus we can take it to have the form

$$\mu(j, \lambda) = (\lambda - v_j)(\lambda\alpha_j + \beta_j). \tag{19}$$

When the above ansätze are substituted into equation (8) we find, at successive powers of λ, the following equations:

$$\lambda^2: \quad A_j(x)^{-2}\alpha_j = h_j(x)^2 - 1 \tag{20}$$

$$\lambda: \quad A_j(x)^{-2}(\beta_j - \alpha_jv_j) = h_j(x)' + 2f_j(x)h_j(x) + v_j(x) \tag{21}$$

$$1: \quad -A_j(x)^{-2}\beta_jv_j = f_j(x)' + f_j(x)^2 + w_j(x). \tag{22}$$

The solutions of this set of equations naturally split into three distinct branches:

1. If $h_j^2 \equiv 1$ then we can take (without loss of generality) $h_j = 1, \alpha_j = 0$ and $\beta_j = 2$, and solve $A_j^{-2} = f_j(x) + \frac{1}{2}v_j(x)$ from (21). The chain equations (10) obtained from this branch will contain $P_{\rm III}$ as a special case.
2. If $h_j \equiv 0$ we can take $\alpha_j = -1, A_j(x) \equiv 1$ and equation (21) then becomes $v_j(x) = \nu_j + \beta_j$. For convenience we set $v_j(x) \equiv 0$ and $\beta_j = -\nu_j$, which implies that $\mu(j, \lambda) = \nu_j^2 - \lambda^2$ and $u_j(\lambda, x) = -\lambda^2 + w_j$ and we therefore obtain the usual dressing chain for the Schrödinger equation [3, 6], after changing notation $\lambda^2 \to \lambda$, $\nu_j^2 \to \nu_j$. This branch contains $P_{\rm IV}$ and $P_{\rm V}$.
3. Finally, in the generic case we choose $\alpha_j = -1$ so as to get $A_j(x)^{-2} = 1 - h_j(x)^2$. Equations (21) and (22) then yield ODEs for h and f. In part II, this branch will be shown to contain $P_{\rm VI}$.

Note also that the $P_{\rm IV}$, $P_{\rm V}$ branch is obtained as a limit $h \to 0$ from the $P_{\rm VI}$ branch. The $P_{\rm III}$ branch can also be obtained as a limit from the $P_{\rm VI}$ branch, but the limit is singular: let

$$h_j = 1 - \varepsilon\big(f_j + \tfrac{1}{2}v_j\big) \qquad \alpha_j = -2\varepsilon \tag{23}$$

and then as $\varepsilon \to 0$ the leading terms from equations (20), (21) yield $A_j^{-2} = f_j + \frac{1}{2}v_j$ and $\beta_j = 2$, respectively.

In all these branches the Painlevé equations are obtained when the derived chain equations are closed (with 'more or less' periodic boundary conditions as we shall see later on) after two, three or four steps. Each time there exists a 'hierarchy' of equations, obtained by closing the chain after a higher number of steps, but we will not discuss these higher order equations in the present paper.

3. The $P_{\rm IV\text{-}V}$ branch in general

3.1. The generic chain equations

The simplest realization of the above scheme is found in the case of the (ordinary) Schrödinger equation $\big(\partial_x^2 + w(x) - \lambda\big)\psi(\lambda, x) = 0$, i.e., for a generalized potential $u_j(\lambda, x)$ in the operator (1) of the type

$$u_j(\lambda, x) = w_j(x) - \lambda. \tag{24}$$

In this case it is well known that the usual Darboux scheme works [22] and that the F_j are actually λ-independent functions $F_j(\lambda, x) = f_j(x)$. More precisely, as explained above, we must choose particular eigenfunctions $\varphi_j(\nu_j)$ of the Schrödinger operator $\partial_x^2 + w_j(x)$ (with eigenvalues ν_j) such that the $f_j(x)$ are expressed as

$$f_j(x) := (\log \varphi_j)_x \tag{25}$$

hence these latter functions have to satisfy the following specialization of relation (9)

$$f_j(x)' + f_j(x)^2 + w_j(x) - \nu_j = 0. \tag{26}$$

Simultaneously we have to satisfy equation (8) and therefore we choose without loss of generality,

$$\mu_j(\lambda) = (\nu_j - \lambda) \qquad A_j(x) = 1. \tag{27}$$

This then yields the standard Darboux transformation for the Schrödinger operator with $\tilde{f}_j \equiv f_j$ (from (5)), $G_j(x) \equiv \partial_x - f_j(x)$ (from (2)) and with a change in the potentials (7) given by

$$w_{j+1} = w_j + 2f_j' \equiv w_j + 2(\log \varphi_j)_{2x}. \tag{28}$$

The resulting chain equation (10) for such Darboux transformations is the well-known *dressing chain* [1]

$$f_j' + f_{j+1}' = f_j^2 - f_{j+1}^2 + \alpha_j \qquad \alpha_j = \nu_{j+1} - \nu_j. \tag{29}$$

The linear problem for the dressing chain [23] follows from the general expressions (12), (13) and (15), (14), subject to (27):

$$[\partial_x - \mathcal{S} - f_j]\psi_j = 0 \tag{30}$$

$$[\mathcal{S}^2 + f_j\mathcal{S} + \mathcal{S}f_j + \nu_j - \lambda]\psi_j = 0. \tag{31}$$

From equation (16) it can be seen that the compatibility condition of this linear system takes the form of a straightforward operator commutation relation $\left[M_j, L_j^d\right] = 0$.

3.2. Generic bilinearization

Note that the initial potential, say w_0, in the sequence (28) never appears in the dressing chains. In fact, it only becomes relevant when actually (re-)interpreting the solutions of these chains—through (25)—as solutions of Schrödinger equations such as (26), which explicitly depend on specific potentials. One instance where the underlying Schrödinger equations are of importance is when one wishes to obtain a Hirota bilinear form for the dressing chain. For, if in the sequence of coupled Schrödinger equations

$$\left(\partial_x^2 + w_j\right)\varphi_j = \nu_j\varphi_j \qquad (j = 1, 2, 3) \tag{32}$$

with potentials transforming as in (28), we parametrize each potential w_j as

$$w_j = 2(\log\omega_{j-1})_{2x}$$

then we obtain a multiplicative transformation rule for the new functions ω_j,

$$\omega_j = \omega_{j-1}\varphi_j.$$

This suggests a parametrization of the eigenfunctions φ_j (32) as the ratio of ω_j and ω_{j-1}. As is well known, the standard bilinearization of such Schrödinger equations is through a ratio of τ-functions. Here, with the benefit of (considerable) hindsight we set

$$\varphi_j = \frac{\tau_j}{\tau_{j-1}}\,\mathrm{e}^{-\hat{\epsilon}x^2/4} \qquad \omega_j = \tau_j\,\mathrm{e}^{-\hat{\epsilon}^2x^4/96}\,\mathrm{e}^{-(j-1/2)\hat{\epsilon}x^2/4} \qquad \nu_j = \hat{\epsilon}(1-j) + \kappa_{j-1} \tag{33}$$

and then (32) transforms into

$$\left(D_x^2 - \hat{\epsilon}xD_x - \kappa_{j-1}\right)\tau_j\cdot\tau_{j-1} = 0 \tag{34}$$

for some constants $\hat{\epsilon}$ and κ_j. (For a definition of the Hirota D-operators and an introduction to their importance in the context of the Painlevé equations we refer to the review paper [25].)

It can be shown that this chain of bilinear equations is nothing but a similarity reduction of the (2 + 1)-dimensional dressing chain associated with the modified KP hierarchy. This entitles us to refer to these τ_j as genuine *tau-functions* in the sense of Sato theory [26–28]; the specific description of this reduction will be addressed in a separate publication [30]. Note that, as pointed out above, the initial tau-function τ_0 (and hence the potential w_0) appears explicitly in the bilinear chain (34).

Below we will see that the system (34) provides a generic bilinearization for all Painlevé equations (and higher order variants) contained in the dressing chain. There do exist other possibilities when it comes to bilinearizing the Painlevé equations, see [25, 29, 30], but we believe the present generic approach to be new. (For a general survey of the Painlevé equations as similarity or symmetry reductions of integrable nonlinear partial differential equations, see, e.g., [31].)

3.3. Periodic closing of the dressing chain

As was mentioned in the introduction we are interested in dressing chains of finite length. Specifically, we will impose *periodic closing conditions* on the sequences of Darboux transformations used in the construction of the chains, i.e., following Adler [6], we require that—up to a shift in the eigenvalues ν_j—the eigenfunctions $\varphi_j(\nu_j)$ which define the Darboux transformations (as in (25)) become periodic with some period $N \geqslant 1$:

$$\varphi_{j+N}(\nu_{j+N}) = \varphi_j(\nu_j) \qquad \text{with} \qquad \nu_{j+N} = \nu_j - \varepsilon \tag{35}$$

for a shift $\varepsilon \neq 0$.

As these functions satisfy $\left(\partial_x^2 + w_j\right)\varphi_j = \nu_j\varphi_j$, we immediately find that the periodic closing of the sequence of functions φ_j also implies that the sequence of potentials w_j (generated through these very Darboux transformations) has to close as

$$w_{j+N}(x) \equiv w_j(x) - \varepsilon \tag{36}$$

with the obvious implication that the sequence of generalized potentials $u_j(\lambda, x)$ (24) closes as

$$u_{j+N}(\lambda, x) \equiv u_j(\lambda + \varepsilon, x) \tag{37}$$

for generic eigenvalues λ.

The closing conditions (35) take on a particularly simple form when expressed on the bilinear chain (34):

$$\tau_{j+N} = \tau_j \qquad \kappa_{j+N} = \kappa_j \qquad \text{and} \qquad \hat{\epsilon} = \frac{\varepsilon}{N}. \tag{38}$$

It should be remarked that as a consequence one also obtains the constraint

$$\prod_{j=1}^{N} \varphi_j = \mathrm{e}^{-\varepsilon x^2/4} \tag{39}$$

from (33). Finally, for the variables (and parameters) which appear in the dressing chains (29), the periodicity conditions take the form

$$f_{j+N} = f_j \qquad \alpha_{j+N} = \alpha_j \qquad \sum_{j=1}^{N} \alpha_j = -\varepsilon \tag{40}$$

accompanied by the first integral (which holds for all $N > 0$)

$$\sum_{j=1}^{N} f_j = -\frac{\varepsilon x}{2}. \tag{41}$$

If N is even there is another integral, obtained by summing every other term of (29):

$$\sum_{j=1}^{N} (-1)^j f_j^2 = -\frac{1}{2} \sum_{j=1}^{N} (-1)^j \alpha_j. \tag{42}$$

Note that the cases $N = 1, 2$ are therefore solvable by quadratures.

4. P_{IV}

As was discovered almost a decade ago [2, 3], the 3-periodic dressing chain (29) is nothing but the P_{IV} equation. We shall now go on to show this explicitly on the dressing chain, after which we shall derive a bilinear representation and a Lax pair for the P_{IV} equation.

4.1. The symmetric form of P_{IV}

Using condition (40) at $N = 3$ we obtain from (29) the chain of equations

$$\begin{cases} f_1' + f_2' = f_1^2 - f_2^2 + \alpha_1 \\ f_2' + f_3' = f_2^2 - f_3^2 + \alpha_2 \\ f_3' + f_1' = f_3^2 - f_1^2 + \alpha_3 \end{cases} \tag{43}$$

with

$$\alpha_1 + \alpha_2 + \alpha_3 = -\varepsilon. \tag{44}$$

Expressed in terms of

$$\begin{aligned} g_1 &= f_1 + f_2 = (\log \varphi_1\varphi_2)_x \\ g_2 &= f_2 + f_3 = (\log \varphi_2\varphi_3)_x \\ g_3 &= f_3 + f_1 = (\log \varphi_3\varphi_1)_x \end{aligned} \tag{45}$$

we get

$$\begin{cases} g_1' = g_1(g_3 - g_2) + \alpha_1 \\ g_2' = g_2(g_1 - g_3) + \alpha_2 \\ g_3' = g_3(g_2 - g_1) + \alpha_3. \end{cases} \tag{46}$$

Following Noumi and Yamada [15, 16] we shall refer to this system as the *symmetric form* of the P_{IV} equation. This particular form of the period 3 dressing chain appears already in [3]. However, it is worth pointing out that (46) is already presented in [32] where it is used to integrate a higher order nonlinear differential equation in terms of the P_{IV} equation.

Because of the constraint (44)—or alternatively, as a consequence of (41)—the system (46) can be integrated once:

$$g_1 + g_2 + g_3 = -\varepsilon x \tag{47}$$

(where a possible integration constant only amounts to a translation in x and can therefore be omitted). Eliminating g_3 we get

$$\begin{cases} g_1' = \alpha_1 - 2g_1g_2 - \varepsilon x g_1 - g_1^2 \\ g_2' = \alpha_2 + 2g_1g_2 + \varepsilon x g_2 + g_2^2. \end{cases} \tag{48}$$

These equations are in Hamiltonian form

$$g_1' = \frac{\partial H}{\partial g_2} \qquad g_2' = -\frac{\partial H}{\partial g_1} \tag{49}$$

where

$$H = -g_1g_2^2 - g_2g_1^2 - \varepsilon x g_1g_2 - \alpha_2 g_1 + \alpha_1 g_2. \tag{50}$$

Note that this Hamiltonian form of P_{IV} differs slightly (by a simple canonical transformation) from the one discussed by Okamoto [7, 9, 18]; it does, however, appear in this form in the classic work [33], and in [15].

Finally, the P_{IV} equation in its standard form is obtained from (48) if we eliminate g_2 and denote $y(z) = \kappa g_1(x)$, where $x = \kappa z$, $\kappa^2 = 2/\varepsilon$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} = \frac{1}{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - a)y + \frac{b}{y} \tag{51}$$

where the parameters a and b are given by

$$a = (\alpha_2 - \alpha_3)/\varepsilon \qquad b = -2(\alpha_1/\varepsilon)^2. \tag{52}$$

4.2. Bilinear form of P_{IV}

Imposing conditions (38) on the bilinear form (34) at $N = 3$ we immediately obtain a bilinearization of the P_{IV} equation:

$$\begin{cases} \left(D_x^2 - \frac{\varepsilon x}{3} D_x - \kappa_0\right)\tau_1 \cdot \tau_0 = 0 \\ \left(D_x^2 - \frac{\varepsilon x}{3} D_x - \kappa_1\right)\tau_2 \cdot \tau_1 = 0 \\ \left(D_x^2 - \frac{\varepsilon x}{3} D_x - \kappa_2\right)\tau_0 \cdot \tau_2 = 0. \end{cases} \tag{53}$$

The transformation for the g_i is

$$g_i = \partial_x \log \frac{\tau_{i+1}}{\tau_{i-1}} - \frac{\varepsilon x}{3} \qquad \alpha_i \equiv \kappa_i - \kappa_{i-1} - \frac{\varepsilon}{3} \tag{54}$$

for $i = 1, 2, 3$, and with periodicity (38) (see also [15, 16]). Recall that (53) was obtained by making the substitution (33) directly into the (now periodic) chain of Schrödinger equations (32). If, on the other hand, we make the substitutions (54) into the symmetric form (46) we will only obtain two equations for the three tau-functions. This is because the equations (46) do not contain any information on the potentials w. However, if we explicitly include the assumption $w_j = 2(\log \tau_{j-1})_{2x}$, e.g., in the form of the first equation of (53), then we do obtain a well-determined system of bilinear equations. Note also that

$$y(z) = \partial_z \log\left(\frac{\tau_2}{\tau_0}\, \mathrm{e}^{-\frac{z^2}{3}}\right). \tag{55}$$

4.3. Lax pair for P_{IV}

Let us see what the consequences of the periodic closing (35) are on the linear problem (30), (31). According to (37) the generalized potentials $u_j(\lambda, x)$ are periodic for generic λ as well (up to a shift in λ) and thus we may require similar periodicity for some eigenfunctions of the linear problem (1), namely those generated by successive Darboux transformations:

$$\psi_{j+3}(\lambda, x) = \psi_j(\lambda + \varepsilon, x). \tag{56}$$

The linear system (30), (31) therefore reduces to the following set of 'difference equations' in the spectral parameter λ:

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_2 + f_1\psi_1 \\ \psi_3 + f_2\psi_2 \\ \psi_1^s + f_3\psi_3 \end{pmatrix} \qquad \begin{pmatrix} \psi_3 + g_1\psi_2 + \nu_1\psi_1 \\ \psi_1^s + g_2\psi_3 + \nu_2\psi_2 \\ \psi_2^s + g_3\psi_1^s + \nu_3\psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{57}$$

where the symbol ψ_j^s denotes eigenfunctions at shifted values of the spectral parameter: $\psi_j^s(\lambda, x) := \psi_j(\lambda + \varepsilon, x)$.

In order to recover a more customary linear problem for the P_{IV} equation, let us first rewrite the equations (57) as

$$\partial_x \Psi(\lambda) = \mathcal{B}_1 \Psi(\lambda) + \mathcal{B}_2 \Psi(\lambda + \varepsilon) \tag{58}$$

$$(\mathcal{A}_1 - \lambda \mathbb{I})\Psi(\lambda) + \mathcal{A}_2 \Psi(\lambda + \varepsilon) = 0 \tag{59}$$

for $\Psi(\lambda) = (\psi_1(\lambda, x), \psi_2(\lambda, x), \psi_3(\lambda, x))^t$ and with matrices $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\begin{aligned} \mathcal{B}_1 &= \begin{pmatrix} f_1 & 1 & 0 \\ 0 & f_2 & 1 \\ 0 & 0 & f_3 \end{pmatrix} & \mathcal{B}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \mathcal{A}_1 &= \begin{pmatrix} \nu_1 & g_1 & 1 \\ 0 & \nu_2 & g_2 \\ 0 & 0 & \nu_3 \end{pmatrix} & \mathcal{A}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ g_3 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{60}$$

If we now define the (formal) Fourier transform of $\Psi(\lambda)$ as

$$\tilde{\Phi}(k) \equiv \int \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}k\lambda}\Psi(\lambda) \tag{61}$$

we immediately see that $\tilde{\Phi}(k)$ satisfies local relations in k, for example,

$$\partial_x \tilde{\Phi}(k) = \mathcal{B}_1\tilde{\Phi}(k) + \mathcal{B}_2\,\mathrm{e}^{-\mathrm{i}\varepsilon k}\tilde{\Phi}(k) \tag{62}$$

instead of the original difference relation (58) in λ. Consequently, introducing a new parameter ξ and a new function $\Phi(\xi)$ in terms of $\tilde{\Phi}(k)$ by

$$\xi := \mathrm{e}^{-\mathrm{i}\varepsilon k} \qquad \Phi(\xi) := \tilde{\Phi}(k)|_{k=\frac{\mathrm{i}}{\varepsilon}\log\xi} \tag{63}$$

we find that $\Phi(\xi)$ satisfies the following Lax pair for the P_{IV} equation,

$$\begin{cases} \partial_x\Phi = \mathcal{M}\Phi & \mathcal{M} \equiv \mathcal{B}_1 + \xi\mathcal{B}_2 \\ -\varepsilon\xi\partial_\xi\Phi = \mathcal{L}\Phi & \mathcal{L} = \mathcal{A}_1 + \xi\mathcal{A}_2 \end{cases} \tag{64}$$

and the compatibility condition $\partial_x\mathcal{L} + \varepsilon\xi\partial_\xi\mathcal{M} = [\mathcal{M}, \mathcal{L}]_-$ is nothing but the symmetric form (46) for P_{IV}. This Lax pair already appears in [16], in connection with reductions of the Drinfeld–Sokolov hierarchy. Also, the transformations (61) and (63) combined, amount to the Mellin transformation that is used (to the same effect) in [21] in order to connect Lax pairs arising in the context of dressing chains to those that appear in the work by Noumi *et al*.

5. P_{V}

Similar to the above, P_{V} can be obtained by closing the dressing chain (29) at $N = 4$ [2, 3]. Just as for the P_{IV} equation this insight yields immediate access to a Lax pair and bilinear formulation for the P_{V} equation.

5.1. Periodic closing and P_{V}

Closing the Darboux chain (29) at period $N = 4$ (cf condition (40)) yields the following system of differential equations:

$$\begin{cases} f_1' + f_2' = f_1^2 - f_2^2 + \alpha_1 \\ f_2' + f_3' = f_2^2 - f_3^2 + \alpha_2 \\ f_3' + f_4' = f_3^2 - f_4^2 + \alpha_3 \\ f_4' + f_1' = f_4^2 - f_1^2 + \alpha_4 \end{cases} \tag{65}$$

with constants α_j ($j = 1, 2, 3, 4$) subject to the constraint

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\varepsilon. \tag{66}$$

Now we have two first integrals

$$f_1 + f_2 + f_3 + f_4 = -\tfrac{1}{2}\varepsilon x \tag{67}$$

$$-f_1^2 + f_2^2 - f_3^2 + f_4^2 = \tfrac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) =: \omega \tag{68}$$

(recall (41), (42)). Hence, the system (65) can be reduced to a second-order differential equation, which turns out to be nothing other than the Painlevé V equation. We shall now proceed to show this.

It is convenient to rewrite the symmetric form (65) in terms of new variables g_1, g_2 obtained by resolving (67) and (68) as

$$\begin{aligned} f_1 &= -\frac{1}{2\varepsilon x} g_1 g_2 + \frac{1}{4}(g_1 - g_2) - \frac{\varepsilon x}{8} + \frac{\omega}{\varepsilon x} \\ f_2 &= \frac{1}{2\varepsilon x} g_1 g_2 + \frac{1}{4}(g_1 + g_2) - \frac{\varepsilon x}{8} - \frac{\omega}{\varepsilon x} \\ f_3 &= -\frac{1}{2\varepsilon x} g_1 g_2 - \frac{1}{4}(g_1 - g_2) - \frac{\varepsilon x}{8} + \frac{\omega}{\varepsilon x} \\ f_4 &= \frac{1}{2\varepsilon x} g_1 g_2 - \frac{1}{4}(g_1 + g_2) - \frac{\varepsilon x}{8} - \frac{\omega}{\varepsilon x} \end{aligned}$$

the inverse relation of which is

$$g_1 = f_1 + f_2 - f_3 - f_4 \qquad g_2 = -f_1 + f_2 + f_3 - f_4$$

together with the constraints (67) and (68). In terms of these new free functions g_1 and g_2 we obtain the symmetric form

$$g_1' = -\frac{1}{\varepsilon x} g_1^2 g_2 + \frac{2\omega}{\varepsilon x} g_1 + \frac{\varepsilon x}{4} g_2 + \alpha_1 - \alpha_3 \tag{69}$$

$$g_2' = \frac{1}{\varepsilon x} g_1 g_2^2 - \frac{2\omega}{\varepsilon x} g_2 - \frac{\varepsilon x}{4} g_1 + \alpha_2 - \alpha_4 \tag{70}$$

which is Hamiltonian (49) with

$$H := -\frac{1}{2\varepsilon x} g_1^2 g_2^2 + \frac{\varepsilon x}{8}\left(g_1^2 + g_2^2\right) + \frac{2\omega}{\varepsilon x} g_1 g_2 - (\alpha_2 - \alpha_4) g_1 + (\alpha_1 - \alpha_3) g_2. \tag{71}$$

Note that this Hamiltonian form of P_{V} differs from the form used in [7, 12, 18, 33], but they are connected by a canonical transformation and a change of the independent variable.

The P_{V} equation is now obtained if we first solve g_2 from (69), substitute it into (70) and express g_1 in terms of y defined by

$$y = \frac{1}{2} - \frac{1}{\varepsilon x} g_1 \qquad \text{where} \qquad y = y(z) \qquad z = \varepsilon x^2/4$$

this yields the P_{V} equation in the form (used, e.g., in [25, 29], up to an extra transformation of the independent variable)

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = &\left(\frac{1}{2y} + \frac{1}{2(y-1)}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}y}{\mathrm{d}z} - \frac{\alpha y}{z^2(y-1)} \\ &- \frac{\beta(y-1)}{z^2 y} - \frac{\gamma y(y-1)}{z} - \delta y(y-1)(2y-1) \end{aligned} \tag{72}$$

where

$$\begin{aligned} \alpha &= \frac{\left(\alpha_1 - \alpha_3 - \omega + \frac{1}{2}\varepsilon\right)^2}{8\varepsilon^2} \qquad \beta = -\frac{\left(\alpha_1 - \alpha_3 + \omega - \frac{1}{2}\varepsilon\right)^2}{8\varepsilon^2} \\ \gamma &= \frac{\alpha_4 - \alpha_2}{\varepsilon} \qquad \delta = -\frac{1}{2}. \end{aligned}$$

The usual form of P_{V} is obtained for $w = y/(y-1)$, which permutes the poles at 1 and ∞.

5.2. Bilinear form of P_V

Imposing conditions (38) on the bilinear form (34) of the dressing chain, this time at $N = 4$, we obtain a bilinearization of the P_V equation in terms of four tau-functions:

$$\begin{cases} \left(D_x^2 - \frac{\varepsilon x}{4} D_x - \kappa_0\right)\tau_1 \cdot \tau_0 = 0 \\ \left(D_x^2 - \frac{\varepsilon x}{4} D_x - \kappa_1\right)\tau_2 \cdot \tau_1 = 0 \\ \left(D_x^2 - \frac{\varepsilon x}{4} D_x - \kappa_2\right)\tau_3 \cdot \tau_2 = 0 \\ \left(D_x^2 - \frac{\varepsilon x}{4} D_x - \kappa_3\right)\tau_0 \cdot \tau_3 = 0. \end{cases} \tag{73}$$

Subject to the same remarks as in the case of the P_{IV} equation, this system of Hirota equations gives a bilinear representation for the symmetric form (65) using substitution

$$f_i = \partial_x \left(\log \frac{\tau_i}{\tau_{i-1}} \right) - \frac{\varepsilon x}{8} \qquad \alpha_i \equiv \kappa_i - \kappa_{i-1} - \frac{\varepsilon}{4} \tag{74}$$

for $i = 1, 2, 3, 4$, and with periodicity (38). Note also that (cf (55))

$$y(z) = \partial_z \log \left(\frac{\tau_0}{\tau_2} \, \mathrm{e}^{\frac{z}{2}} \right). \tag{75}$$

It was already mentioned that in [14] Noumi and Yamada presented a class of dynamical systems, each member of which possesses a particular symmetry group of (affine) Weyl-type $\widehat{W}\left(A_{n-1}^{(1)}\right)$. P_{IV} and P_V are contained in this class at levels $n = 3$ and 4, respectively. It can be shown [30, 34], not only that the bilinear forms of the P_{IV} and P_V equations (53) and (73) possess similar symmetries, but generally that *any* bilinear system contained in (34) possesses a symmetry group of type $\widehat{W}(A_{n-1}^{(1)})$. Hence these equations are nothing but the bilinear formulations of the Noumi–Yamada systems and consequently, we have shown that each member in that class corresponds to a periodic dressing chain (in the sense of sections 3.2 and 3.3).

5.3. Lax pair for P_V

Exactly as in the P_{IV} case one can obtain a Lax pair for P_V by imposing periodicity on the generic Lax pair (30), (31) for the dressing chain:

$$\psi_{j+4}(\lambda, x) = \psi_j(\lambda + \varepsilon, x). \tag{76}$$

Following the notation of section 4.3 we can write

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_2 + f_1\psi_1 \\ \psi_3 + f_2\psi_2 \\ \psi_4 + f_3\psi_3 \\ \psi_1^s + f_4\psi_4 \end{pmatrix} \qquad \begin{pmatrix} \psi_3 + (f_1 + f_2)\psi_2 + \nu_1\psi_1 \\ \psi_4 + (f_2 + f_3)\psi_3 + \nu_2\psi_2 \\ \psi_1^s + (f_3 + f_4)\psi_4 + \nu_3\psi_3 \\ \psi_2^s + (f_4 + f_1)\psi_1^s + \nu_4\psi_4 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{77}$$

After a (formal) Fourier transformation similar to that in the P_{IV} case (61), (63), we obtain the following Lax formulation for the P_V equation [16]:

$$\partial_x \Phi = \mathcal{M}\Phi \qquad -\varepsilon\xi\partial_\xi \Phi = \mathcal{L}\Phi \tag{78}$$

$$\mathcal{M} = \begin{pmatrix} f_1 & 1 & 0 & 0 \\ 0 & f_2 & 1 & 0 \\ 0 & 0 & f_3 & 1 \\ 0 & 0 & 0 & f_4 \end{pmatrix} + \xi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{79}$$

$$\mathcal{L} = \begin{pmatrix} \nu_1 & f_1+f_2 & 1 & 0 \\ 0 & \nu_2 & f_2+f_3 & 1 \\ 0 & 0 & \nu_3 & f_3+f_4 \\ 0 & 0 & 0 & \nu_4 \end{pmatrix} + \xi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ f_1+f_4 & 1 & 0 & 0 \end{pmatrix}. \tag{80}$$

The compatibility condition $\partial_x \mathcal{L} + \varepsilon\xi\partial_\xi \mathcal{M} = [\mathcal{M}, \mathcal{L}]_-$ is easily seen to correspond to the symmetric form (65) of the P_{V} equation.

6. The P_{III} branch in general

As was noted in the introduction, the P_{III} branch is quite different from that for the P_{IV}–P_{V} equations. In this section we will first study this branch in general and in the next section we shall close the chain in order to obtain P_{III}.

6.1. The generic chain equations

As was explained in section 2.3, a different branch of the chain equation (10) for potentials of type (17) is obtained if we choose

$$F_j(\lambda, x) := \lambda + f_j(x) \tag{81}$$

$$\mu_j(\lambda) := 2(\lambda - \nu_j) \tag{82}$$

$$A_j(x) := \left[f_j(x) + \tfrac{1}{2}v_j(x)\right]^{-\frac{1}{2}} \tag{83}$$

$\left(f_j + \frac{1}{2}v_j \not\equiv 0\right)$; the only remaining part of condition (8) is the following constraint on f (22):

$$f_j' + f_j^2 + \nu_j(2f_j + v_j) + w_j = 0. \tag{84}$$

For the present transformation to be a genuine Darboux transformation, we must also require that the intertwiner G_j in (2) annihilates some particular eigenfunction $\varphi_j := \psi(\nu_j, x)$ of the original Schrödinger equation, i.e., $F_j(\nu_j, x) = (\log\varphi_j)_x$ or in terms of f:

$$f_j := -\nu_j + \varphi_j'/\varphi_j. \tag{85}$$

Condition (84) then turns into the Schrödinger equation for the eigenfunction φ_j:

$$\varphi_j'' + \left(-\nu_j^2 + \nu_j v_j + w_j\right)\varphi_j = 0. \tag{86}$$

The transformation induced for the potentials is obtained from (7):

$$v_{j+1} = v_j + 2(\log A_j)' \tag{87}$$

$$w_{j+1} = w_j + [2f_jA_j - A_j']'/A_j. \tag{88}$$

From the explicit form (83) of A it is clear that in the present case the chain equations generated by (10)—subject to condition (84)—will take the form of a dynamical system in the variables f and v. However, these variables are strongly coupled and from (84), (87), (88) we get (after eliminating w)

$$f_j' + \tfrac{1}{2}v_j' + \left(f_j + \tfrac{1}{2}v_j\right)(v_{j+1} - v_j) = 0 \tag{89}$$

$$f_j' - \tfrac{1}{2}v_j' - (f_{j-1} + \nu_{j-1})^2 + \left(\tfrac{1}{2}v_{j-1} - \nu_{j-1}\right)^2 + (f_j + \nu_j)^2 - \left(\tfrac{1}{2}v_j - \nu_j\right)^2 = 0. \tag{90}$$

It is convenient to change to new dependent variables d_j, r_j defined by

$$f_j = \tfrac{1}{2}d_j + \tfrac{1}{2}r_j - \nu_j \qquad v_j = d_j - r_j + 2\nu_j \qquad \beta_j = \nu_j - \nu_{j+1} \tag{91}$$

(note that $d_j = f_j + \frac{1}{2}v_j \equiv A_j^{-2}$) because then the *chain equations of the $P_{\rm III}$ branch* take the following simple form (which already appears in [4])

$$d_j{}' = d_j(d_j - r_j - d_{j+1} + r_{j+1} + 2\beta_j) \tag{92}$$

$$r_j{}' = d_{j-1}r_{j-1} - d_j r_j. \tag{93}$$

The linear problem for this integrable system is of course obtained from (12), (13) and (15), (14) by appropriate identifications of the variables appearing in those equations, but we shall not discuss its general form here.

6.2. Generic bilinearization

It is of interest to bilinearize the chain equations (92), (93) before closing the chain. We start by representing A in terms of a new function $\tilde{\varphi}_j$ and the eigenfunction φ_j:

$$A_j^{-2} = \frac{\tilde{\varphi}_j}{\varphi_j} \tag{94}$$

which due to (85) amounts to

$$\varphi_j' = \tilde{\varphi}_j + \left(v_j - \tfrac{1}{2}v_j\right)\varphi_j. \tag{95}$$

Since φ_j is a solution of (86) we also obtain

$$\tilde{\varphi}_j' = \left(\tfrac{1}{2}v_j - v_j\right)\tilde{\varphi}_j + \left(\tfrac{1}{2}v_j' - w_j - \tfrac{1}{4}v_j^2\right)\varphi_j. \tag{96}$$

Taken together, equations (95), (96) yield a first-order representation of the Schrödinger equation (86) and as such provide a suitable starting point for the bilinearization of the Schrödinger equations associated with the $P_{\rm III}$ branch.

Under the above ansatz, the transformation formula (87) suggests a logarithmic parametrization for the potential v

$$v_j := (\log q_j)' \tag{97}$$

because then the transformation for q takes the simple multiplicative form

$$q_{j+1} \sim q_j\varphi_j/\tilde{\varphi}_j \tag{98}$$

where $\sim$ denotes equivalence up to a constant multiple. The transformation formula for w is best expressed in terms of a new variable W

$$w_j = \tfrac{1}{2}W_j - \tfrac{1}{4}v_j^2 \tag{99}$$

because then we get

$$W_{j+1} = W_j + (\log\varphi_j)'' - (\log q_j)'' + (\log\tilde{\varphi}_j)'' \tag{100}$$

(from (88) using also (95), (98)) which suggests a parametrization

$$W_j := (\log\omega_j)''. \tag{101}$$

The transformation for W then takes the simple multiplicative form

$$\omega_{j+1} \approx \frac{\omega_j\varphi_j\tilde{\varphi}_j}{q_j} \sim \frac{\omega_j\varphi_j^2}{q_{j+1}} \tag{102}$$

(where $\approx$ denotes equivalence up to a gauge factor e^{ax+b}).

If, as before, we think of the functions φ_j, $\tilde{\varphi}_j$, q_j and ω_j as ratios of 'tau-functions', it becomes clear that

$$q_j = \tau_j^+ \big/ \tau_j \, \mathrm{e}^{\gamma_j x} \qquad \omega_j = \tau_j \tau_j^+ \tag{103}$$

$$\varphi_j = c_j \frac{\tau_{j+1}^+}{\tau_j^+} \sqrt{\frac{\tau_j^+}{\tau_j}} \, \mathrm{e}^{\alpha_j x} \qquad \tilde{\varphi}_j = \tilde{c}_j \frac{\tau_{j+1}}{\tau_j} \sqrt{\frac{\tau_j^+}{\tau_j}} \, \mathrm{e}^{\tilde{\alpha}_j x} \tag{104}$$

is a good representation, since it automatically satisfies the multiplicative transformation rules for v and w, equations (98) and (102), if the introduced constants satisfy

$$\gamma_{j+1} = \gamma_j + \alpha_j - \tilde{\alpha}_j. \tag{105}$$

For f and v we get from (81) and (97)

$$f_j = \frac{1}{2} \partial_x \log \left(\frac{{\tau_{j+1}^+}^2}{\tau_j \tau_j^+} \right) + \alpha_j - \nu_j \qquad v_j = \partial_x \log \left(\frac{\tau_j^+}{\tau_j} \right) + \gamma_j. \tag{106}$$

and then

$$d_j = \partial_x \log \left(\frac{\tau_{j+1}^+}{\tau_j} \right) + \alpha_j + \frac{1}{2}\gamma_j - \nu_j \qquad r_j = \partial_x \log \left(\frac{\tau_{j+1}^+}{\tau_j^+} \right) + \alpha_j - \frac{1}{2}\gamma_j + \nu_j. \tag{107}$$

As mentioned before, the assignments (103), (104) take care of equations (87), (88), but we still have to study the Schrödinger equation (86), which we take in its first-order form (95), (96). This yields

$$\begin{cases} \big(D_x + \alpha_j - \nu_j + \frac{\gamma_j}{2}\big)\tau_{j+1}^+ \cdot \tau_j = \frac{\tilde{c}_j}{c_j} \, \mathrm{e}^{(\tilde{\alpha}_j - \alpha_j)x} \tau_j^+ \tau_{j+1} \\ D_x\big(D_x + \alpha_j - \nu_j + \frac{\gamma_j}{2}\big)\tau_{j+1}^+ \cdot \tau_j = \frac{\tilde{c}_j}{c_j} \, \mathrm{e}^{(\tilde{\alpha}_j - \alpha_j)x} (D_x + \gamma_j - \alpha_j - \tilde{\alpha}_j - 2\nu_j)\tau_j^+ \cdot \tau_{j+1}. \end{cases}$$

In (103), (104) we have some additional freedom in the parameters $\gamma, \alpha, \tilde{\alpha}$ (by scaling $\tau_j^+ \to \tau_j^+ \, \mathrm{e}^{s_j x}$, $\tau_j \to \tau_j \, \mathrm{e}^{-s_j x}$), which can be used to assign a fixed value to the parameter γ_j

$$\gamma_j = 2(\nu_j - \alpha_j). \tag{108}$$

In order to satisfy (105) we must then also require that

$$\tilde{\alpha}_j = 2\alpha_{j+1} - \alpha_j - 2(\nu_{j+1} - \nu_j). \tag{109}$$

After this redefinition of parameters, the equations simplify considerably and we get

$$\begin{cases} D_x \tau_{j+1}^+ \cdot \tau_j = \kappa_j \, \mathrm{e}^{\sigma_j x} \tau_j^+ \tau_{j+1} \\ D_x^2 \tau_{j+1}^+ \cdot \tau_j = \kappa_j \, \mathrm{e}^{\sigma_j x} (D_x - \rho_j)\tau_j^+ \cdot \tau_{j+1}. \end{cases} \tag{110}$$

where

$$\begin{aligned} \sigma_j &= \tilde{\alpha}_j - \alpha_j = 2(\alpha_{j+1} - \alpha_j - \nu_{j+1} + \nu_j) \\ \rho_j &= \tilde{\alpha}_j + 3\alpha_j = 2(\alpha_{j+1} + \alpha_j - \nu_{j+1} + \nu_j) \\ \kappa_j &= \tilde{c}_j / c_j. \end{aligned}$$

Equations (110) form the Hirota bilinear representation of the chain equations (92), (93). For later use note that due to the first equation of (110) we can also write

$$d_j = \kappa_j \, \mathrm{e}^{\sigma_j x} \frac{\tau_j^+ \tau_{j+1}}{\tau_{j+1}^+ \tau_j}. \tag{111}$$

6.3. Periodic closing in general

As before, we are mainly interested in a finite chain of equations obtained after requiring the potential $u_j(\lambda)$ to be periodic, up to a shift in the (generic) eigenvalue, as stated in (37), and this implies the existence of 'periodic' eigenfunctions (35). For the potentials v, w this now implies

$$v_{j+N} = v_j - 2\varepsilon \qquad w_{j+N} = w_j + \varepsilon v_j - \varepsilon^2 \tag{112}$$

whereas the auxiliary potential W of (99) is strictly periodic: $W_{j+N} = W_j$.

On the level of the τ-functions τ, τ^+ these closing conditions translate into strict periodicity

$$\tau^+_{j+N} = \tau^+_j \qquad \tau_{j+N} = \tau_j. \tag{113}$$

Condition (105) was replaced by constraints (108) and (109), which are compatible with the requirement that the parameters $\alpha_j, \tilde{\alpha}_j, c_j, \tilde{c}_j, \sigma_j, \rho_j, \kappa_j$ are strictly periodic and

$$\gamma_{j+N} = \gamma_j - 2\varepsilon. \tag{114}$$

We also have

$$\sum_{j=1}^{N} \beta_j = \varepsilon \qquad \sum_{j=1}^{N} \sigma_j = 2\varepsilon. \tag{115}$$

Furthermore, from (107), (108) it is obvious that d_j and r_j are (strictly) periodic as well and from (111) and (107) we also find the following two (generic) conservation laws for the periodic chain equations [4]:

$$\prod_{j=1}^{N} d_j = \mathrm{e}^{2\varepsilon x} \prod_{j=1}^{N} \kappa_j \qquad \sum_{j=1}^{N} r_j = 2 \sum_{j=1}^{N} \alpha_j. \tag{116}$$

7. P_{III}

7.1. Closing the chain at $N = 2$

Due to the existence of two conservation laws the system obtained from (92), (93) at $N = 1$ is of course trivial. However, at $N = 2$ we find that d_1, d_2, r_1, r_2 satisfy the four basic equations

$$d_1' = d_1(d_1 - d_2 + r_2 - r_1 + 2\beta_1) \tag{117}$$

$$d_2' = d_2(d_2 - d_1 + r_1 - r_2 + 2\beta_2) \tag{118}$$

$$r_1' = d_2 r_2 - d_1 r_1 \tag{119}$$

$$r_2' = d_1 r_1 - d_2 r_2 \tag{120}$$

with $\beta_1 + \beta_2 = \varepsilon$. We shall refer to this system as the 'symmetric form of P_{III}'. We now show how P_{III} is obtained from it.

Two first integrals were given in (116)

$$d_1 d_2 = \kappa_1 \kappa_2 \, \mathrm{e}^{2\varepsilon x} \qquad r_1 + r_2 = \tfrac{1}{2}(\rho_1 + \rho_2) - \varepsilon. \tag{121}$$

We resolve these by introducing new functions g_i defined as follows:

$$\begin{aligned} &d_1 = \kappa_1 g_1 \, \mathrm{e}^{\varepsilon x} \qquad d_2 = \kappa_2 g_1^{-1} \, \mathrm{e}^{\varepsilon x} \\ &r_1 = -g_1 g_2 + \tfrac{1}{2}(\rho_1 + \rho_2) - \varepsilon \qquad r_2 = g_1 g_2. \end{aligned} \tag{122}$$

Then the equations (117)–(120) take the form

$$g_1' = 2g_1^2 g_2 + \mathrm{e}^{\varepsilon x}\kappa_1 g_1^2 - (\rho_2 - \varepsilon)g_1 - \mathrm{e}^{\varepsilon x}\kappa_2 \tag{123}$$

$$g_2' = -2g_1 g_2^2 - 2\,\mathrm{e}^{\varepsilon x}\kappa_1 g_1 g_2 + (\rho_2 - \varepsilon)g_2 + \mathrm{e}^{\varepsilon x}\kappa_1\big(\tfrac{1}{2}(\rho_1 + \rho_2) - \varepsilon\big) \tag{124}$$

which is Hamiltonian with

$$H = g_1^2 g_2^2 + \mathrm{e}^{\varepsilon x}\kappa_1 g_1^2 g_2 - (\rho_2 - \varepsilon)g_1 g_2 - \mathrm{e}^{\varepsilon x}\big[\kappa_1\big(\tfrac{1}{2}(\rho_1 + \rho_2) - \varepsilon\big)g_1 + \kappa_2 g_2\big]. \tag{125}$$

This Hamiltonian form of P_{III} is the same as that given in [7, 13, 18], up to a simple transformation.

If we solve g_2 from (123), substitute it into (124), and use the new variables

$$y(z) = \sqrt{\frac{\kappa_1}{\kappa_2}}\,g_1(x) \qquad \text{where} \qquad z = \frac{\sqrt{\kappa_1\kappa_2}}{\varepsilon}\,\mathrm{e}^{\varepsilon x} \tag{126}$$

we get P_{III} in the canonical form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}y}{\mathrm{d}z} + y^3 + \frac{1}{z}\left(\frac{\rho_1}{\varepsilon}y^2 - \frac{\rho_2}{\varepsilon}\right) - \frac{1}{y}. \tag{127}$$

7.2. Bilinear form of P_{III}

The bilinear form of P_{III} is basically nothing other than (110) with periodic τ. Since the τ appear in pairs (τ_1, τ_2^+ and τ_2, τ_1^+) and since bilinear equations are gauge invariant, we propose the following gauge transformation and scaling

$$\begin{aligned} &\tau_1 = \sqrt{\kappa_1}\,\bar\tau_1\,\mathrm{e}^{\theta x} \qquad \tau_2^+ = \bar\tau_2^+\,\mathrm{e}^{\theta x} \qquad \tau_1^+ = \bar\tau_1^+ \\ &\tau_2 = \sqrt{\kappa_2}\,\bar\tau_2 \qquad \theta = \tfrac{1}{4}(\sigma_1 - \sigma_2) \end{aligned} \tag{128}$$

which yield, after a subsequent x-translation $x \to x - \frac{1}{2\varepsilon}\log(\kappa_1\kappa_2)$, the following novel bilinear form of P_{III}:

$$D_x\bar\tau_2^+\cdot\bar\tau_1 = \mathrm{e}^{\varepsilon x}\bar\tau_1^+\bar\tau_2 \tag{129}$$

$$D_x\bar\tau_1^+\cdot\bar\tau_2 = \mathrm{e}^{\varepsilon x}\bar\tau_2^+\bar\tau_1 \tag{130}$$

$$D_x^2\bar\tau_2^+\cdot\bar\tau_1 = \mathrm{e}^{\varepsilon x}(D_x - \rho_1)\bar\tau_1^+\cdot\bar\tau_2 \tag{131}$$

$$D_x^2\bar\tau_1^+\cdot\bar\tau_2 = \mathrm{e}^{\varepsilon x}(D_x - \rho_2)\bar\tau_2^+\cdot\bar\tau_1. \tag{132}$$

From this P_{III} can also be derived directly: comparing (111), (122), (126) (after the aforementioned gauge transformation) one can solve for $\bar\tau_2$ in terms of y and the other $\bar\tau$. Then from (129) one solves for $\partial_x\bar\tau_1$ and from (130) for $\partial_x\bar\tau_2^+$. Then (131)$\times y$−(132) is nothing but the P_{III} equation (127). In these new variables y and z we also have (cf (55), (75))

$$y(z) = \partial_z \log\frac{\bar\tau_2^+}{\bar\tau_1}. \tag{133}$$

It should be noted that bilinear forms are sensitive to changes in the independent variable. For example, if we express the bilinear system obtained after the transformation (128) (but without the x-translation), in terms of the z-coordinate (now regarding the tau-functions $\bar\tau_2^+$, τ_1, $\bar\tau_1^+$, τ_2 as functions of z) we get a system of equations which is no longer expressible in Hirota D-operators only as it also involves ordinary z-derivatives. However, the system so obtained can be shown to be a reduction of an integrable system contained in the so-called 'modified 2-component KP hierarchy' (at least in the restricted case where $\rho_2 = \pm\rho_1$) and hence the functions $\bar\tau_2^+$, τ_1, $\bar\tau_1^+$, τ_2 introduced here can be thought of as genuine tau-functions in the sense of Sato theory.

7.3. Lax pair for P_{III}

Just as for P_{IV}, a Lax pair for the Painlevé III equation can be derived from the linear formulation (13), (14) introduced earlier for the general chain equations (10).

As before, the standard periodic closing (35) involves a shift in the eigenvalues ν_j associated with the eigenfunctions φ_j (for the P_{III} case one has $\nu_3 = \nu_1 - \varepsilon$). Hence, from the condition $\varphi_3(\nu_3) = \varphi_1(\nu_1)$ and expression (85) it follows that $f_3 = f_1 + \varepsilon$. Furthermore, due to the relation $u_3(\lambda) = u_1(\lambda + \varepsilon)$ there will exist eigenfunctions of the Schrödinger equation (1) for which $\psi_3(\lambda, x) = \psi_1(\lambda + \varepsilon, x)$ holds. This then, bearing in mind relation (81) and the fact that $A_3 = A_1$ (actually $A_3^2 = A_1^2$; we choose the sign $A_3 = A_1$), yields the following periodic closing of the linear equations (13)

$$\partial_x \psi_1 = A_1^{-1}\psi_2 + (\lambda + f_1)\psi_1 \tag{134}$$

$$\partial_x \psi_2 = A_2^{-1}\psi_1^s + (\lambda + f_2)\psi_2 \tag{135}$$

and (14)

$$(A_1A_2)^{-1}\psi_1^s + A_1^{-1}(2\lambda + f_1 + f_2 - (\log A_1)')\psi_2 + 2(\lambda - \nu_1)A_1^{-2}\psi_1 = 0 \tag{136}$$

$$(A_1A_2)^{-1}\psi_2^s + A_2^{-1}(2\lambda + \varepsilon + f_1 + f_2 - (\log A_2)')\psi_1^s + 2(\lambda - \nu_2)A_2^{-2}\psi_2 = 0 \tag{137}$$

where ψ_i^s stands for $\psi_i(\lambda + \varepsilon, x)$.

It is now advantageous to change to scaled eigenfunctions

$$\hat{\psi}_i := A_i^{-1/2}\, \mathrm{e}^{(-\varepsilon - 2(\nu_1+\nu_2)+r_1+r_2)x/4}\psi_i$$

(recall that $r_1 + r_2$ is a constant). After also changing to the new independent variable $\mathfrak{z} = \varepsilon z = \sqrt{\kappa_1\kappa_2}\,\mathrm{e}^{\varepsilon x}$ and using the previously obtained formulae we can write (134)–(137) as

$$\varepsilon\mathfrak{z}\partial_{\mathfrak{z}}\hat{\psi}_1 = \tfrac{1}{4}(4\lambda + d_1 + d_2 + \varepsilon - 4\nu_1 + 4\nu_2 + 2r_1 - 2r_2)\hat{\psi}_1 + \sqrt{\mathfrak{z}}\hat{\psi}_2 \tag{138}$$

$$\varepsilon\mathfrak{z}\partial_{\mathfrak{z}}\hat{\psi}_2 = \tfrac{1}{4}(4\lambda + d_1 + d_2 - \varepsilon + 4\nu_1 - 4\nu_2 - 2r_1 + 2r_2)\hat{\psi}_2 + \sqrt{\mathfrak{z}}\hat{\psi}_1 \tag{139}$$

$$\mathfrak{z}\hat{\psi}_1^s + \sqrt{\mathfrak{z}}(2\lambda + d_1 + r_2 - 2\nu_2)\hat{\psi}_2 + 2d_1(\lambda - \nu_1)\hat{\psi}_1 = 0 \tag{140}$$

$$\mathfrak{z}\hat{\psi}_2^s + \sqrt{\mathfrak{z}}(2\lambda + d_2 + r_1 - 2\nu_1 + 2\varepsilon)\hat{\psi}_1^s + 2d_2(\lambda - \nu_2)\hat{\psi}_2 = 0. \tag{141}$$

In exactly the same way as for the P_{IV} equation, this system of difference equations (in the spectral parameter) can be cast into a more standard form by introducing the formal Fourier transform of the eigenfunctions $\psi_j(\lambda, x)$:

$$\tilde{\Phi}_j(k, x) := \int \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}k\lambda}\hat{\psi}_j(\lambda, x). \tag{142}$$

In terms of the new variable $\xi = \exp(-\mathrm{i}\varepsilon k)$ and the new dependent variables $\Phi_j(\xi, \mathfrak{z}) := \tilde{\Phi}_j(k, x)$, we then obtain the linear systems

$$\varepsilon\xi\partial_\xi \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \frac{1}{2}\left[D_1 + A + \frac{1}{\xi - \mathfrak{z}}B\right]\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \tag{143}$$

$$\varepsilon\mathfrak{z}\partial_{\mathfrak{z}} \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \frac{1}{2}\left[D_2 + A - \frac{1}{\xi - \mathfrak{z}}B\right]\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \tag{144}$$

where

$$A = \begin{pmatrix} \nu_2 - \nu_1 & \sqrt{\delta} \\ \xi\sqrt{\delta} & \nu_1 - \nu_2 \end{pmatrix} \qquad B = \begin{pmatrix} \xi r_1 & -d_2 r_2/\sqrt{\delta} \\ -\xi d_1 r_1/\sqrt{\delta} & \xi r_2 \end{pmatrix} \tag{145}$$

$$\begin{aligned} D_1 &= -(\nu_1 + \nu_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ D_2 &= \left[\frac{1}{2}(d_1 + d_2) + \nu_1 + \nu_2\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left[\frac{1}{2}\varepsilon + r_1 - r_2\right]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{146}$$

The compatibility of these two matrix equations yields the equations (117)–(120).

Although our result is also expressed in terms of 2×2 matrices this Lax pair is quite different from the classical one given in [8], which is commonly used when discussing isomonodromy problems in connection with P_{III}. It also seems to be unrelated to linear systems that would result from a similarity reduction of the linear triads for the Loewner–Konopelchenko–Rogers system, a system that incorporates the Einstein–Weyl equations that possess a similarity reduction to the (full parameter) P_{III} equation [35].

8. Conclusions

In this paper we have constructed Darboux chains from (scalar) Schrödinger equations for generic second-order energy-dependent potentials and then proceeded to discuss periodic reductions of such chains. The Darboux chains were classified into three types, the so-called $P_{\mathrm{IV-V}}$, P_{III} and P_{VI} branches, which derive their names from the Painlevé equations that arise as the lowest period (non-trivial) reductions contained in each of them. (A detailed discussion of the P_{VI} branch will be given in a sequel to the present paper.)

We described in detail the construction of the generic chain equations (29) and (92), (93) for the $P_{\mathrm{IV-V}}$ and P_{III} branches, and their bilinearization (34), (110). For the periodically closed chains we described the reduction to the corresponding Painlevé equations, with an intermediate equation in Hamiltonian form (see also [4]).

The bilinearization of the generic Darboux chains in the $P_{\mathrm{IV-V}}$ and P_{III} branches led to a representation of the eigenfunctions in terms of tau-functions (33), (104), which in reduction gave rise to bilinear representations and tau-function formulae for the $P_{\mathrm{III-V}}$ equations (53), (73), (129)–(132) and their solutions (55), (75), (133). From the bilinear form of the $P_{\mathrm{IV-V}}$ branch it is clear that the equations described by it are identical to the so-called $A_n^{(1)}$-type dynamical systems introduced by Noumi and Yamada [14]. The exact nature of the tau-functions associated with the periodic reductions of this branch in the context of Sato theory will be discussed in a forthcoming paper (see [34] for a discussion of the P_{IV} case). The precise link between the tau-functions that appear in the periodic reductions of the P_{III} branch and Sato theory is currently being investigated.

The construction of the Darboux chains presented in this paper also allowed us to systematically derive Lax representations for the chain equations and their reductions, resulting in a novel Lax description of the P_{III} equation (143)–(146). The Lax pairs obtained for the $P_{\mathrm{IV-V}}$ branch were again related to those obtained by Noumi and Yamada for the $A_n^{(1)}$-type dynamical systems mentioned before.

The interpretation of the Hamiltonians associated with the periodic reductions of the Darboux chains, in terms of the tau-functions that describe their solutions, as well as a detailed investigation of the higher order members—i.e., those corresponding to chains with higher periods—in the different branches of chain equations, are topics that will be addressed in subsequent papers.

Finally we should comment on the two Painlevé equations that have not been discussed in this paper: the $P_{\rm II}$ and $P_{\rm I}$ equations. As we only treated the Darboux transformations for a general (energy-dependent) Schrödinger equation, we had to omit the $P_{\rm II}$ equation from our analysis. In principle, one should be able to obtain it by closing a Darboux chain at step two but for a second-order spectral problem the result would be trivial and one has to resort to something else. Indeed, in [3] the $P_{\rm II}$ equation arises from a chain of Darboux transformations for a third-order spectral problem (although admittedly a special one that still allows for Darboux transformations similar to those for the Schrödinger equation). For the $P_{\rm I}$ equation the situation is again different. The fact that certain Painlevé equations arise from periodic Darboux chains is intimately related to the existence of an affine Weyl group of symmetries for these equations. The $P_{\rm I}$ equation does not posses such symmetries and hence does not arise from a periodic Darboux chain (it is actually related to stationary reductions of integrable systems [6]).

Acknowledgments

The authors would like to thank Professor A Veselov for discussions. JH would like to thank J Satsuma for warm hospitality during a visit to the Tokyo University Graduate School of Mathematical Sciences. The authors have also benefited from discussions and correspondence with R Conte, M Noumi and K Takasaki. This work was partially supported by a grant-in-aid from the Japan Society for the Promotion of Science (JSPS).

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